

# Spherical black holes with regular center

— a review of existing models including a recent realization with Gaussian sources —

Stefano Ansoldi\*

*Department of Physics, Kyoto University, Kyoto, Japan  
International Center for Relativistic Astrophysics, Pescara, Italy  
Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, Italy*

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## Abstract

We review, in a historical perspective, some results about black hole spacetimes with a regular center. We then see how their properties are realized in a specific solution that recently appeared; in particular we analyze in detail the (necessary) violation of the strong energy condition.

## 1 Introduction

Recently there has been a renewed interest for the search of solutions of Einstein equations, mostly motivated by the study of higher dimensional gravity, for instance in the context of the brane world scenario [104, 105] and of string theory [110]. Some beautiful examples of higher dimensional solutions have appeared (see, e.g., [68, 67]), which feature interesting properties that, in the pure 4-dimensional framework, are absent. This renewed interest is also having some influence on a rather more specialized, but very interesting, research area, that of regular black holes. It is of course well-known that, under rather generic conditions on the energy-matter content of spacetime, classical solution of Einstein equations exhibit both, future [99] and past [60, 61, 62, 51, 63] singularities [65, 31, 14, 114, 90, 129, 64] usually hidden by an event horizon [71]. This fact, which is clearly exemplified by the first and probably most well-known solution of Einstein equations, i.e. the Schwarzschild solution [112, 111], has been likely fully appreciated only after the study of its global analytic extension [44, 46, 74, 125, 95, 90, 64], which gives a consistent picture of its properties,

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\*Email: [ansoldi@trieste.infn.it](mailto:ansoldi@trieste.infn.it) — Webpage: <http://www-dft.ts.infn.it/~ansoldi>

like the existence of the region inside the horizon, in which the radial and time coordinate exchange their character, and the presence of the central singularity. Although the presence of, both, the black hole region and the central singularity has been eventually accepted, i.e. we learned how to classically live with them, especially the presence of a singularity is a recurrent motivation to underline the inadequacy of general relativity as a theory of spacetime below some length scale (apart from the theoretical motivations, it is *a fact* that, experimentally, gravity can be tested only in a finite range of scales). On one side, maybe the most well known one, this has motivated the search for a more complete theory of gravity including also quantum effects (see, e.g., [73] for a recent, comprehensive review). On the other side, it has also sustained many efforts to push as much as possible Einstein gravity to its limit, trying to avoid, if not the black hole region, at least the central singularity in a way as consistent as possible with physical requirements<sup>1</sup>. Following some very early ideas that date back to the work of Sakharov [109], Gliner [53] and Bardeen [9], solutions having a global structure very similar to the one of black hole spacetimes, but in which the central singularity is absent, have been found (references to them *will appear* in the rest of the paper).

In this contribution we are going to briefly review some of these ideas, but, before concluding this introductory section with the layout of the rest of the paper, we would like to make a couple of remarks. The first of them is that, in contrast to the fact that nowadays the world of theoretical physics witnesses a consistent number of strongly believed (but as yet unproven) conjectures, most of the results about black holes and their properties are, in fact *theorems*. Theorems usually make some hypotheses, which in this case can be roughly interpreted within a threefold scheme; i) the validity of some geometric properties, usually related to the behavior of geodesics (for instance the existence of trapped surfaces, and so on); ii) the validity of some conditions on the matter fields that are coupled to gravity (energy conditions); iii) the validity of some (more technical) hypotheses about the global/causal structure of spacetime. It is then clear that the possibility of singularity avoidance, within the context defined by general relativity, requires the violation of, at least, one of the above conditions. Since conditions of type iii) are mostly technical, there has been a great effort to make them as general as possible (although sometimes this means less immediate) if not to remove them at all, by generalizing the earliest results [65, 64, 129]. Conditions i) are usually the requirement that some *indicator* exists, which emphasizes that something a little bit unusual for a “flat” non covariant mind is taking place in spacetime, and are usually related to the existence of horizons, so there is little reason to modify them. It is then natural that, as a possible way to avoid singularities, a relaxation of conditions of type ii) has been advocated. With a strongly conservative attitude, a word of caution should be sounded at this point. It is, in fact, known that matter and energy violating some of the energy conditions, have *as yet* unobserved proper-

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<sup>1</sup>The two points of view just outlined should be seen as complementary and, often, even integrated.

ties<sup>2</sup>: this means that we are *not yet* able to produce them in a laboratory by a well-known, generally reproducible procedure. We have, nevertheless, good candidates to realize these violations when we treat at an effective level the quantum properties of spacetime and matter at some length/energy scales: this is very suggestive, since it directly connects to the, possibly, incomplete character of classical general relativity as a theory of spacetime and with the ongoing, diversified, efforts toward its quantization [73]. To review in more detail some aspects related to the above reflections, we plan as follows.

In section 2 we review various regular models of spacetime, centering our attention, almost exclusively, on regular black holes of a very specific type (specified below). After a review of the earliest ideas (subsection 2.1) we analyze their first concrete (and, perhaps, to some extent independent) realization, known as the Bardeen solution: we review also some studies, which appeared much later, discussing its global character (subsection 2.2); we then continue with a discussion of black hole interiors (subsection 2.3) reporting various early proposals, which adopted spacetime junctions to get rid of singularities; this brings us to the central part of this section (subsection 2.4), where some exact solutions are analyzed, together with the possibility of physical realizations for the energy-matter content which should act as their source (subsubsection 2.4.1). The solutions that we will have described up to this point are not extemporary realizations, but can be understood in a very interesting, complete and general framework: we thus review the essence of this framework in subsection 2.5. This section is concluded with a very concise summary of the results that we have reviewed (subsection 2.6).

Then, in section 3 we use a recently obtained solution, which is another possible realization of the general type of solutions described in subsection 2.5, to perform a simple exercise, i.e. the study of the violation of one of the energy conditions. For completeness, after introducing the algebraic form of the solution, we quickly construct its global spacetime structure in subsection 3.1 (this result follows immediately from the results reviewed in subsection 2.5); we then show which regions of spacetime are filled with matter violating the strong energy condition (subsection 3.2). The results of this second part of the paper are summarized in subsection 3.3.

Some general comments and remarks find space in the concise concluding section, i.e. section 4. We now conclude this introduction by fixing one notation and one naming convention, as below.

## 1.1 Conventions and notations

In what follows we will concentrate on spherically symmetric solutions of Einstein equations and restrict ourself to media which satisfy the condition that the radial pressure equals the opposite of the energy density. We will then use,

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<sup>2</sup>We will come back to this point later on, mentioning vacuum and the cosmological constant. A clear discussion of this point can be found in the standard reference [64]; see also the early [107] for a physically oriented discussion of the implications of a violation of the weak energy condition.

throughout and unless otherwise stated, the coordinate system  $(t, r, \vartheta, \varphi)$ , in which the metric can be written in the static form adapted to the spherical symmetry, i.e.

$$g_{\mu\nu} = \text{diag} \left( -f(r), f(r)^{-1}, r^2, r^2 \sin^2 \vartheta \right). \quad (1)$$

As apparent from the above definition we adopt the signature  $(-, +, +, +)$ . We occasionally will use the name *metric function* for the function  $f(r)$ . We do not spend extra comments about the meaning of the coordinate choice, which is standard and discussed in detail in various textbooks (see for instance [90]; any other textbook choice will be equivalent); Thus, without restating every time our coordinate choice, in what follows we will specify various metrics just by specifying the corresponding metric function. In view of the above, when we will have to discuss the maximal extension of solutions that admit an expression of the metric in the form (1), although we will follow the naming conventions of the standard reference [64] for boundaries as infinity, only in one point of our discussion we will need a few more global ideas than the one concisely and effectively discussed in [130]. We will moreover use the standard notation  $T_{\mu\nu}$  for the stress-energy tensor which appears on the righthand side of Einstein equations.

## 2 Regular spacetime models

As we briefly discussed in the introductory section, there are various reasons to try to find consistent solutions of Einstein equations that describe regular spacetimes. As we also briefly discussed, the possibilities are somewhat restricted by the existence of various singularity theorems [99, 60, 61, 62, 51, 63, 65, 31, 14, 114, 90, 64, 129], which apply, both, to cosmology as well as to gravitational collapse. Thus, the real crucial point is if/how it is possible to violate some of the hypothesis of singularity theorems, without obtaining unphysical models. The literature developed a lot in this direction, in the field of, both again, cosmology and gravitational collapse. Solutions that do not posses horizons are, of course, a natural framework in which singularity theorems can not be applied and in which regular solutions can be searched: boson stars are an example (see [72, 108] as well as, e.g., [89, 128] for recent reviews). On the other hand it seems natural that in various physically reasonable situations horizons are indeed formed<sup>3</sup>. It becomes, thus, interesting to consider if singularities can be avoided in presence of horizons. Solutions for which this is true are generically called *regular black hole*. A great variety of these solutions has been investigated (a paper presenting a broad perspective on the subject is [45]) and a very effective classification of them can be found in [24]: we are going to follow this classification scheme and concentrate our attention on existing models of *black*

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<sup>3</sup>The problem of the nature of the horizon and of the inside horizon region is a fascinating one, which has been carefully analyzed in the past (see, for instance, [71, 100, 101, 97, 13, 102] and references therein) also in presence of a non-abelian hair [32], i.e. for the so called Einstein-Yang-Mills black holes (additional references can be found in the bibliography of the already cited [32]).

*holes with a regular center*, i.e. the type 1 regular black holes in the above mentioned classification (see [24], page 975). For reasons of space we will also remain in the more standard arena in which only the gravitational field and the electromagnetic field (including non-linear electrodynamics) are present<sup>4</sup>.

## 2.1 The earliest ideas

In the above defined context, the first seeds that will eventually lead to the idea of regular black hole solutions go back to the mid sixties with Sakharov proposal to consider the equation of state (pressure) =  $-(\text{energy density})$  as the appropriate equation of state for matter and energy at very high densities [109]. This equation of state is the same equation of state obeyed by the *cosmological vacuum*, by which we are going to identify the cosmological constant term of Einstein equation, when we decide to interpret it as a contribution to the right hand side of Einstein equations (i.e. to the energy momentum tensor), rather than as a, left hand side, geometric entity. It can be agreed to a great extent that, as very briefly, but precisely, noted in [34] (footnote on page 738), this may seem a rather philosophical distinction between *a vacuum spacetime* and a *spacetime filled with vacuum*; on the other hand, we would like to stress a related physical point of view i.e. the fact that the first one, a *a vacuum spacetime* where the cosmological term is a geometrical parameter, does satisfy all the known energy conditions, whereas the second one, i.e. the one *filled with vacuum*, does not satisfy the strong energy condition. Thus, in the second case, the related question of the physical origin of the vacuum, i.e. of a medium having a negative pressure equal in magnitude to its energy density (we are, of course, making the safe assumption  $\rho > 0$ ), becomes a relevant one. We will, briefly, come back to this point later on. Nevertheless, it is clearly very interesting to investigate the consequences of accepting (physically and not only philosophically) the idea of a *spacetime filled with vacuum*, hence the early proposal of Gliner [53] that a *spacetime filled with vacuum* could provide a proper description of the final stage of gravitational collapse, replacing the future singularity<sup>5</sup>. This proposal, which should have helped getting rid of the singularity in favor of a regular spacetime, was not immediately realized by a specific solution of Einstein equations. In general a solution of this kind would require a consistent treatment of, both, the gravitational fields and the matter fields *at the quantum level*. We are still waiting for a framework capable to provide us with this treatment; nevertheless already the classical analysis turns out to be quite interesting, as shown for instance by the seminal paper [28].

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<sup>4</sup>Nevertheless, we would like to stress that studies of more general situations, for instance in presence of Yang-Mills fields [122, 19] and of massive [29], nonlinear [22, 50] and phantom [27] scalar fields do exist. Generalizations (see for instance [119] and [115]) of existing models (the ones in [55] and [33], respectively) witness an interest for higher dimensional extensions, which nowadays can be motivated by various studies, e.g. those about brane-world black holes [126, 23, 75, 127].

<sup>5</sup>The term used by Gliner, which corresponds to our *spacetime filled with vacuum*, is  $\mu$ -vacuum [53].

## 2.2 The Bardeen solution

This said, it is then interesting to note that, just a few years later the appearance of these seminal ideas and likely with different motivations, the first regular solution of Einstein equations having an event horizon was obtained by J. M. Bardeen [9]. Although this was not realized for some time, this solution is a concrete example of a completely general class of models (discussed below) that realize singularity avoidance using a vacuum-like equation of state for matter below some length scale. Bardeen's solution (see also [14]) is a solution of Einstein equations in the presence of an electromagnetic field and, as the well-known Reissner-Nordström solution [106, 94], is parametrized by mass  $m$  and charge  $e$ . Also the line element can be put in exactly the same form, i.e the static and spherically symmetric one defined in (1), with the metric function  $f(r)$  being

$$f(r) = 1 - \frac{2mr^2}{(r^2 + e^2)^{3/2}} = 1 - \left(\frac{m}{e}\right) \frac{2(r/e)^2}{((r/e)^2 + 1)^{3/2}}$$

and  $r \geq 0$  (notice the equal sign in the previous relation). It is not difficult to see that i) there are values of the  $m/e$  ratio for which  $f(r)$  has no zeroes (and is thus always positive), ii) values for which it has two zeroes, as well as an in between case, where iii) the function is always non-negative but vanishes at only one point together with its first derivative. Notice that this function is everywhere well defined as are the curvature tensors and scalar of the solution that it represents; the metric described by  $f(r)$  is asymptotically flat, for small  $r$  it behaves as the de Sitter metric, since

$$f(r) \approx 1 - 2\frac{m}{e}r^2, \quad r \approx 0^+,$$

whereas for large  $r$  it asymptotically behaves as the Schwarzschild metric. In the case ii) mentioned above the black hole interior does not terminate on a singularity but crosses an interior Cauchy horizon and develops in a region that becomes more and more de Sitter like, eventually ending with a regular origin at  $r = 0$ . This solution represents a strong constraint on possible generalizations of existing singularity theorems, since it acts as a counterexample that prevents relaxing too much the necessary conditions for the existence of a singularity. It, also, has very interesting properties and its causal structure and its geometrical structure have been studied in detail [14, 15], although many years after its appearance. From these studies, it turned out that there can be a very generic mechanism for singularity avoidance in classical general relativity, which requires what has been called a *topology change*. If we, again, restrict our attention to the values of the parameters for which the Bardeen solution has both an event and a Cauchy horizon, it can be easily seen that its global structure is as in figure 1, i.e. it resembles the maximal extension [57, 30] of Reissner-Nordström [106, 94] spacetime but with a regular origin. The analysis of Borde [14, 15], then, shows that the singularity avoidance can be related to the fact that in regions where spacelike slices of the Reissner-Nordström solution would have the topology of

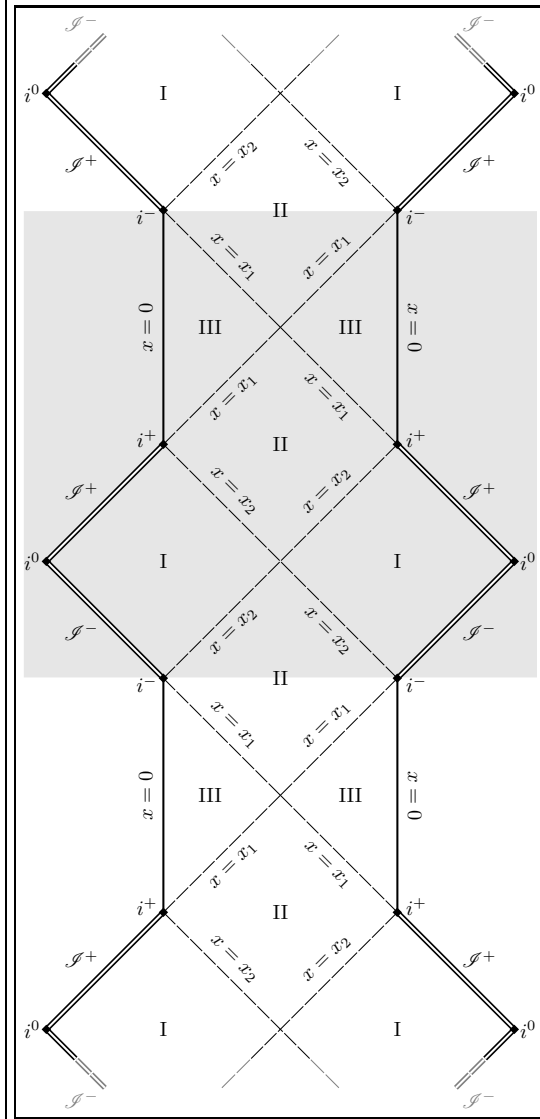


Figure 1: maximal extension of the metric of a black hole with a regular center when, both, the event and the Cauchy horizons are present. Variable  $x$  is a rescaled, dimensionless radial coordinate, obtained rescaling  $r$  by a characteristic length scale of the solution. Labels of the boundaries of spacetime follow the notation in [64]. The maximal extension is Resiřner-Nordstrřm-like, but  $x = 0$  is non-singular. Each point in the diagram corresponds to a 2-sphere  $\mathbb{S}_2$ . The double lines represent future and past null infinity, whereas the dashed ones correspond to values of  $x$  at which the metric function vanishes, i.e. they are the event horizon ( $x = x_2$ ) and the Cauchy horizon ( $x = x_1$ ). Light rays lines form a 45 degree angle with the horizontal direction. From the diagram it is readily seen that the spacetime is not globally hyperbolic and consists of an infinite sequence of regions, some in which the metric function is positive and some in which the metric function is negative (in particular the regions bounded by  $x = 0$  and those bounded by null infinity are of the first kind, whereas the ones

bounded by  $x = x_1$  and  $x = x_2$  are of the second kind). As discussed in the main text, this maximal extension is a completely general results for metrics which satisfy the conditions discussed on page 14 in subsection 2.5, when the parameters of the model fall in some specific range (for the metric in (4) and (5) this range is  $\mu_0 > \mu_0^{\text{cr}}$ ). The way in which this solution avoids the emergence of singularities is related to the structure of the regions between the two *antipodal*  $x = 0$  (i.e.  $r = 0$ ) lines: here future directed causal curves can *wrap around the universe* until they finally free themselves crossing a type II region and ending up in an asymptotically flat type III region (see also the main text, starting at page 6 in subsection 2.2).

$\mathbb{R} \times \mathbb{S}^1$ , the same spacelike slices in the Bardeen solution have the topology  $\mathbb{S}^3$ . Since there are corresponding regions of the extended manifold where spacelike slices of both, the Reißner-Nordström and the Bardeen solution have, indeed, the topology  $\mathbb{R} \times \mathbb{S}^1$ , in [14, 15] the avoidance of singularity has been associated to the *topology change* which appears in the structure of the spacelike slices, which from open become closed; this allows singularity avoidance<sup>6</sup> since in the closed case

it is possible for light rays to “wrap around the Universe;” i.e., although both the “ingoing” and “outgoing” systems of future-directed null geodesics [*starting\**] from [*a trapped surface\**]  $\mathcal{T}$  are converging, the two sets converge to focii at different  $r = 0$  points (antipodally located with respect to each other) [*quotation taken from* [15]; *\*present author addition*].

## 2.3 Black hole interiors

The Bardeen solution represents a first concrete case that realizes the early physical idea of Sakharov [109] and Gliner [53], replacing the singularity by a regular de Sitter core. In view of the early appearance of the Schwarzschild [112, 111] and de Sitter [121, 120] solutions and of the global structure of these spacetimes, it is natural that the idea to replace the black hole interior of Schwarzschild spacetime with the interior (i.e., before the cosmological horizon) region of de Sitter spacetime appeared quite early [55] (with related thermodynamical interpretations [56]) together with charged generalizations [117, 116]; this intuitive ideas can be supported, in fact, by the useful formalism of Israel junction conditions [69, 70, 11]. Despite the fact that the simpler idea to perform the junction at a null surface [118] fails due to stability issues [59] and the appearance of a discontinuity in the pressure at the null junction [58, 102], it is, instead, possible to substitute *part* of the black hole interior with part of de Sitter spacetime interposing a layer of non-inflationary material [58]. It is particularly interesting the case in which this layer is spacelike [47, 48], which gives the so called Schwarzschild–de Sitter model<sup>7</sup>. A motivation for these studies is naturally the fact that, approaching a singularity, the curvature becomes unbounded. On the other hand and quite generically, a consistent framework for singularity avoidance (due, e.g., to quantum [49, 91] or, nowadays more fashionable and exotic, quintessential [87] effects or motivated by a theory more fundamental than general relativity [73]) should prevent this divergence<sup>8</sup>, so that it is rather natural to assume an upper bound for the curvature [78, 79, 80, 81, 82, 83], which in these early approaches was naturally taken at the Planck scale. Under this hypothesis, it can be shown that the black hole region of the Schwarzschild metric can

<sup>6</sup>This provides an earlier answer to the much later question recently raised in [88].

<sup>7</sup>Not to be confused with the Schwarzschild–de Sitter solution of Einstein equations.

<sup>8</sup>Similar ideas can, of course, be applied to the initial singularity in cosmology and we will touch briefly this point later on; meanwhile, we would like to remember some interesting proposals [124, 92, 18], also related to what we are going to discuss right below. A discussion in connection with wormholes also appeared [103].



be joined along a spacelike junction to de Sitter space, the transition happening in a time of the order of the Planck time [47, 48]. The procedure can be carried out not only between the eternal, static spacetimes, but also considering the black hole as a result of gravitational collapse; moreover, it is possible for the de Sitter space to decay into a Friedman universe, from which the suggestive idea that a new universe might be created from gravitational collapse arises. Concentrating a little bit more on the technical aspects of this early proposal, in view of our future discussion we would like to point out that, because of the spacelike character of the junction, the resulting spacetime contains Cauchy horizons and, thus, is not globally hyperbolic; we also remark, that the spacelike hypersurface at which the Kasner-like contraction of the Schwarzschild collapse is turned into the deflation of de Sitter spacetime is followed by another hypersurface, where the transition to an inflationary space takes place. This last surface is topologically a three sphere  $\mathbb{S}^3$ , so that the new world formed inside the black hole is actually a closed one (see references quoted above). This not a coincidence, if we remember our discussion of the Bardeen solution on page 6 in subsection 2.2. We also emphasize, for future reference, that the global structure of spacetime substantially corresponds to the light-grayed area of the maximal extension of the Bardeen solution of figure 1. It is also interesting to observe that it can be proved [8] that this model is stable in the following sense: inside the black hole region the  $T_t^t$  component of the stress energy tensor can be interpreted as a tension along the axis of a three-cylinder of constant time  $r = \text{const.}$ ; this is true, in particular, along the surface at which the junction is performed and it can be seen that there are values of the parameters for which fluctuations of the Schwarzschild mass  $M$  and/or of the de Sitter cosmological constant  $\Lambda$  and/or of the other internal parameters, as the surface pressures, do not induce a collapse of the three cylinder to zero radius, but give rise, instead, to spatial oscillations of its radius with a longitudinal dependence. In a modern perspective, it is also interesting to remember a later extension of this early model [10], discussing the possibility of creation of multiple de Sitter universes with null boundaries inside the black hole region, a picture very similar to the one of an eternally inflating universe.

Other kinds of matching, which do not involve the presence of a surface layer, can also be performed: in [84] the mass function is given in two explicit examples, where the junctions are performed at the horizon of Schwarzschild spacetime, which represents the exterior asymptotically flat region of the solutions. Complete extensions of the manifold are also discussed and they cannot be obtained by analytical continuation, which is not possible across the horizon [86]. As discussed in [84], it is also possible to perform the junction away from the horizon without changing the general properties of the result: still the space-time manifold, which can be extended to completeness, is not analytical, so that its extension is not unique. More complicated models with similar properties can also be obtained.

## 2.4 Exact solutions

All the above models, with the exception of the Bardeen solution, are not exact solutions of Einstein equations in the usual sense, since they do not satisfy everywhere the required analyticity properties of the sources and/or the field equations. It is thus interesting to consider if the Bardeen solution is just an isolated counterexample or can be obtained as a particular case in a more general class of solutions. We will review, in the following the instructive path followed in the literature to prove that the latter applies. Seminal ideas in this direction (in the framework of cosmology) actually appeared very early and much closer in time to the original Bardeen solution than to related future developments: in particular already in the mid seventies a *regular* cosmological model was presented, where the idea that, with the increase of density matter would enter a state dominated by a negative pressure [54], was realized. In this model a huge increase in the mass of the universe from the epoch of the beginning of expansion was obtained, a result that suggestively anticipates some ideas that will be developed only years later, when regular cosmological models will be formulated. Closer in spirit to the idea of curing future singularities is, instead, the exact solution of Einstein equations presented in [33]. In the notation introduced above in (1), this spherically symmetric static solution of Einstein equations is characterized by

$$f(r) = 1 - \frac{m(r)}{r}, \quad \text{where} \quad m(r) = r_g \left( 1 - \exp \left( -\frac{r^3}{r_*^3} \right) \right); \quad (2)$$

this model has, in fact, two free parameters which, in view of their physical meaning it is convenient to choose as  $r_g$  and  $r_0 = \sqrt{r_*^3/r_g}$ . It is not difficult to check that if  $r \gg r_*$  the metric resembles better and better the Schwarzschild solution, whereas if  $r \ll r_*$  a de Sitter like behavior is recovered [33], since the metric behaves as

$$f(r) \approx 1 - \frac{r_g}{r_*^3} r^2 = 1 - \frac{r^2}{r_0^2} \quad \text{when} \quad r \approx 0.$$

The stress-energy tensor acting as source of this solution is such that  $T_0^0 = T_1^1 \neq T_2^2 = T_3^3$  (the last equality is a consequence of spherical symmetry). Following [53] we see that in this model the source of the gravitational field is a realization of what can be called a radial vacuum. It can be seen that the above model is regular everywhere; on the other hand, for some values of the parameters an event horizon appears (which, as we will see later, is then paired with a Cauchy horizon), so that energy conditions have to be violated somewhere, and in fact, the strong energy condition is certainly violated at small  $r$ , where the solution becomes de Sitter like. The weak energy condition is instead satisfied everywhere. This fact shows that, when the parameters allow for the presence of an event horizon in this regular spacetime, the spacetime cannot be globally hyperbolic. A detailed analysis of the analytic extension in these situations [41] confirms this educated guess (the global structure of the solution is, again, as in

figure 1, showing an alternating sequence of asymptotically flat Schwarzschild-like regions and regular de Sitter cores separated by black/white hole regions). In view of the fact that there exist extremal configurations in which the Cauchy horizon degenerates with the event horizon, it becomes particularly interesting to analyze the thermodynamic properties of the solution<sup>9</sup>; it is then possible to see that the temperature drops to zero when the two horizons merge and the issue of the stability of the extremal configuration becomes then of utmost importance to understand the final fate of the process [41, 38] (the stability of G-lumps is also being studied [39]). Although the solution described above is asymptotically flat, i.e. it realizes a transition from a non-zero value of the cosmological constant at the center to a zero value at infinity, asymptotically de Sitter generalizations can also be constructed [40]: they are, of course, relevant for cosmological applications<sup>10</sup>. Also this family of solutions, as the one of the black hole type, can be interpreted in a “dynamical cosmological constant” framework [42], but we will not put the accent on this point in what follows. We think that, instead, it can be interesting to remark the fact that the same solution, but in a different range of the relevant parameters, also describes regular spacetimes without horizons, which are called G-lumps and represent a vacuum, self-gravitating, regular, particle-like structure [34, 39], similar in the spirit to boson stars [72, 108], recently reviewed in [89, 128]). A representation of the global structure of these solutions is given in figure 2. In this interpretative framework it is found that the Arnovitt-Deser-Misner (ADM) mass is related to the cosmological term [34], a result which remembers the remark we made at the very beginning of this subsection and that is suggestive of interesting speculations.

#### 2.4.1 Matter content

The results summarized above can be understood in a consistent theoretical framework, which we will discuss more precisely later on, in subsection 2.5. Before that, in this subsection we would like to review how it is possible to find physical motivations for matter content that generates solutions of the kind that we have discussed in the previous subsections. The importance to provide a physical origin for the source term that, *via* Einstein equations, is then associated to regular black holes solutions, has been emphasized since the earliest models that have appeared. We already discussed the very sound physical motivation behind Sakharov [109] and Gliner [53] ideas. Moreover the specific form of the solution (2) has been motivated (see [41] and the references quoted there in the discussion on page 532) using the Schwinger formula [113], according to which the probability  $P_{\text{Schwinger}}$  to find a particle as a result of

<sup>9</sup>Interest for thermodynamic properties of regular spacetimes with horizons appeared early [56], especially in connection with the idea that a remnant of some kind is left at the end of the evaporation stage. See, for instance, [66] and the references discussed in section 3 for recent examples.

<sup>10</sup>See, for instance, [25, 26] and reference therein; we also, incidentally, note that in this context, the formation of baby universes inside regular black holes has been also considered [43].

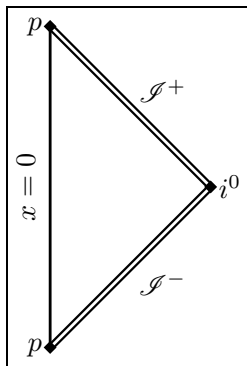


Figure 2: generic spacetime structure for a G-lump. In various regular solutions this structure arises from the same metric that gives rise to the extension in figure 1 but in a different range of the parameters (for instance, with reference to the discussion in subsection 3.1, this structure it is obtained from the metric in (4) and (5) when  $0 < \mu_0 < \mu_0^{\text{cr}}$ ). As before, conventions for names of boundaries follow the ones in [64]. We remember that spacetime is everywhere regular and that it approaches de Sitter spacetime at small  $r$  and Schwarzschild spacetime at large  $r$ .

vacuum polarization in a given field  $\Psi$  can be approximated by

$$P_{\text{Schwinger}} \sim \exp(-\Psi_{\text{cr.}}/\Psi).$$

Using de Sitter spacetime with horizon radius  $r_0$  (see above) to describe the properties of the vacuum with polarization effects that manifest themselves as gravitational field tension, we have

$$\Psi \sim \frac{r_g}{r^3} \quad \text{and} \quad \Psi_{\text{cr.}} = \frac{1}{r_0^2} :$$

in term of a parameter  $r_g$  with the dimension of a length,  $\Psi$  is the curvature term characteristic of the gravitational tension and  $\Psi_{\text{cr.}}$  is the curvature critical value. Thus

$$P_{\text{Schwinger}} \sim \exp(-r^3/(r_0^2 r_g)),$$

which behaves as the  $T_0^0$  associated with the solution (2) of Einstein equations.

In view of the above analogy, it is interesting to consider more elaborate models, in which it is possible to find some self-consistent solutions for the dynamics of gravity and of the fields coupled to it, without assigning the form of the stress-energy tensor *a priori*<sup>11</sup>. A framework in which the above program could be realized, is the one of nonlinear electrodynamics (see<sup>12</sup> Born [16] and

<sup>11</sup>In a historical perspective, the interested reader can take a look at the ideas present in [28], where, in a not too different conceptual context, a similar problem was early investigated.

<sup>12</sup>We emphasize the authors' names in this particular case only, since in some references existing in the literature there is a recurrent imprecision in the list of the authors of [16].

Born and Infled [17]) coupled to gravity. With the above motivation regular solutions in nonlinear electrodynamics theories coupled to gravity have been searched [96, 123, 98]. One example is given by the metric defined by the metric function [3]

$$f(r) = 1 - \frac{2mr^2}{(r^2 + q^2)^{3/2}} + \frac{q^2 r^2}{(r^2 + q^2)^2}$$

for which the electric field, correspondingly, is obtained as

$$E(r) = qr^4 \left( \frac{r^2 - 5q^2}{(r^2 + q^2)^4} + \frac{15}{2} \frac{m}{(r^2 + q^2)^{7/2}} \right);$$

this result is reminiscent of the Bardeen solution [6], of which it shares most of the properties. In view of the discussion in the next section, we especially stress the asymptotic behaviors of the solution, i.e.

$$f(r) \approx 1 - \left( \frac{2m}{q} - 1 \right) \frac{r^2}{q^2}, \quad E(r) \approx \frac{5r^4}{q^5} \left( \frac{3}{2} \frac{m}{q} - 1 \right) \quad \text{when } r \approx 0$$

and

$$f(r) \approx 1 - \frac{2m}{r} + \frac{q^2}{r^2}, \quad E(r) \approx \frac{q}{r^2} \quad \text{when } r \approx +\infty;$$

we see that, again, a de Sitter-like core occupies the small radius region, whereas at infinity the solutions approaches the Reissner-Nordström solution, with the standard  $r^{-2}$  behavior of the electric field:  $m$  and  $q$  can then be interpreted as the mass and charge of the solution and there are again values of the parameters for which the solution, which satisfies the weak energy solution, is regular everywhere [3] but has two horizons: in this case the maximal extension again is given by the diagram in figure 1. Different realizations have also been found [4, 5, 20]. A subsequent general analysis of these solutions was also performed [21] and it was rigorously proved that regular electrically charged structures are not compatible with the Maxwell weak-field limit at the center; regular magnetic solutions are also analyzed in [21], under very general terms, as a viable alternative that respects more closely some standard properties of the classical electromagnetic field. A careful discussion of this and related issues can be found also in [36], where another solution is introduced and its properties in relation with photon propagation and the cosmological constant are analyzed. Another multi-parametric solution was recently discussed in [7] and the spinning case has also been studied [37]. All these solutions have the same/similar general properties of the ones that we have discussed with some more detail.

## 2.5 The general framework

In the previous section we have seen (maybe in a not comprehensive, but still quite wide, perspective) various solutions of Einstein equations that, despite the fact that they have event horizons, are not affected by the presence of singularities. We remember that we have restricted our attention to the solutions which,

in the classification given in [24], are called *black holes with a regular center*. It turns actually out that these solutions can be (and have been) discussed under a very general framework: it has, indeed, been proved that their nature is essentially *unique*, although their specific realizations may analytically differ. We would like to discuss here in more detail these interesting aspects, which relate the properties of the metric functions, with those of the matter sources and the energy conditions that they satisfy. The constraints that have to be satisfied by regular black hole solutions arise from singularities theorems, as well as from other specific results (see, for instance, [12, 85]). In particular, if we consider the case in which the matter is characterized by its energy density  $\rho$ , its radial pressure  $p_r$  and its tangential pressure  $p_t$  (the presence of only one non-radial pressure follows once we restrict ourself to the spherically symmetric case, as we are doing here) it can be proved that, even in presence of junctions, the condition

$$\rho + p_r + 2p_t \geq 0$$

assures the absence of event horizons if the spacetime metric is  $C^2$  everywhere [85]: note that continuity of  $\rho$  and  $p_t$  is not required. Thus, if we are interested in spacetimes which are regular but have an event horizon, the condition written just above, must, in general, be violated. It is very instructive to see that, in fact, black hole solutions with a regular center can be characterized in a *quasi*-complete way. This has been done in [34] (since the analysis in [34] is very clear and detailed we report here only the final result, referring the reader to the original paper for the interesting details of the proof, additional results and a careful discussion). In particular it can be proved that if i) the dominant energy condition is satisfied<sup>13</sup>, ii) the metric is regular at the center (and everywhere else), iii) the energy density is regular at the center, iv) the solution is asymptotically flat with finite ADM mass  $M$  and v) the weak energy condition<sup>14</sup> is satisfied in some region around the origin  $r = 0$ , then  $T_t^t = T_r^r$  and

$$f(r) = 1 - \frac{2m(r)}{r}, \quad m(r) = 4\pi \int_0^r x^2 \rho(x) dx \quad \text{with} \quad \lim_{r \rightarrow +\infty} m(R) = M < \infty;$$

$$\text{moreover } \frac{d\rho}{dr} \leq 0, \quad p_r = -\rho \quad \text{and} \quad p_t = -\rho - \frac{r}{2} \frac{d\rho}{dr} \quad \text{which, in turn, imply}$$

that the solution behaves asymptotically as de Sitter space when  $r \rightarrow 0$ . A straightforward corollary of the above results [34] is that,  $f(r)$  has a maximum

<sup>13</sup>The dominant energy conditions requires  $T^{00} \geq |T^{ab}|$  ( $a, b = 1, 2, 3$ ); this condition is equivalent to [64]  $\rho \geq 0 \wedge -\rho \leq p_k \leq +\rho$  ( $k = 1, 2, 3$ ), where  $\rho$  is the energy density and  $p_k$  ( $k = 1, 2, 3$ ) the principal pressures; in spherical symmetry this is furthermore equivalent to  $\rho \geq 0 \wedge -\rho \leq p_r \leq +\rho \wedge -\rho \leq p_t \leq +\rho$ , where  $p_r$  is the radial pressure and  $p_t$  the tangential pressure.

<sup>14</sup>The weak energy condition requires  $T_{\mu\nu}\xi^\mu\xi^\nu \geq 0$  for any timelike vector  $\xi^\mu$ ; this condition is equivalent to [64]  $\rho \geq 0 \wedge \rho + p_k \geq 0$  ( $k = 1, 2, 3$ ), where  $\rho$  is the energy density and  $p_k$  ( $k = 1, 2, 3$ ) the principal pressures; in spherical symmetry this is equivalent to  $\rho \geq 0 \wedge \rho + p_r \geq 0 \wedge \rho + p_t \geq 0$ , where  $p_r$  is the radial pressure and  $p_t$  the tangential pressure. Note that the weak energy condition is implied by the dominant energy condition, although not equivalent to it.

at  $r = 0$  and can have at most one more extremum that must be a minimum: then the metric can at most have two zeroes, so at most two horizons. This happens when the minimum is negative; when the minimum is exactly zero the two horizons degenerate and when the minimum is positive there are no horizons. The possible causal structures of this class of solutions are reported in figures 1, 2 and 3. We remark the following points.

1. The dominant energy conditions is required only to obtain the regularity of pressures from the assumption of the regularity of the energy density. If regularity of the pressures is instead postulated, it is enough to assume the weak energy condition. In this sense, the weak energy condition singles out the class of regular spherically symmetric metrics, which are asymptotically de Sitter at the center and asymptotically Schwarzschild at infinity; the results about the behavior of the metric function are also a consequence of the validity of the weak energy condition [34].
2. In view of the general results discussed above the matter content associated to this class of solutions describes an anisotropic fluid since  $p_r = p_t$  can be satisfied if and only if  $\rho = \text{const.}$ ; this implies that the only way to have isotropic solutions with a finite ADM mass is to consider vacuum solutions. Moreover, as seen before [85], these solutions will generically have

$$\rho + p_r + 2p_t < 0$$

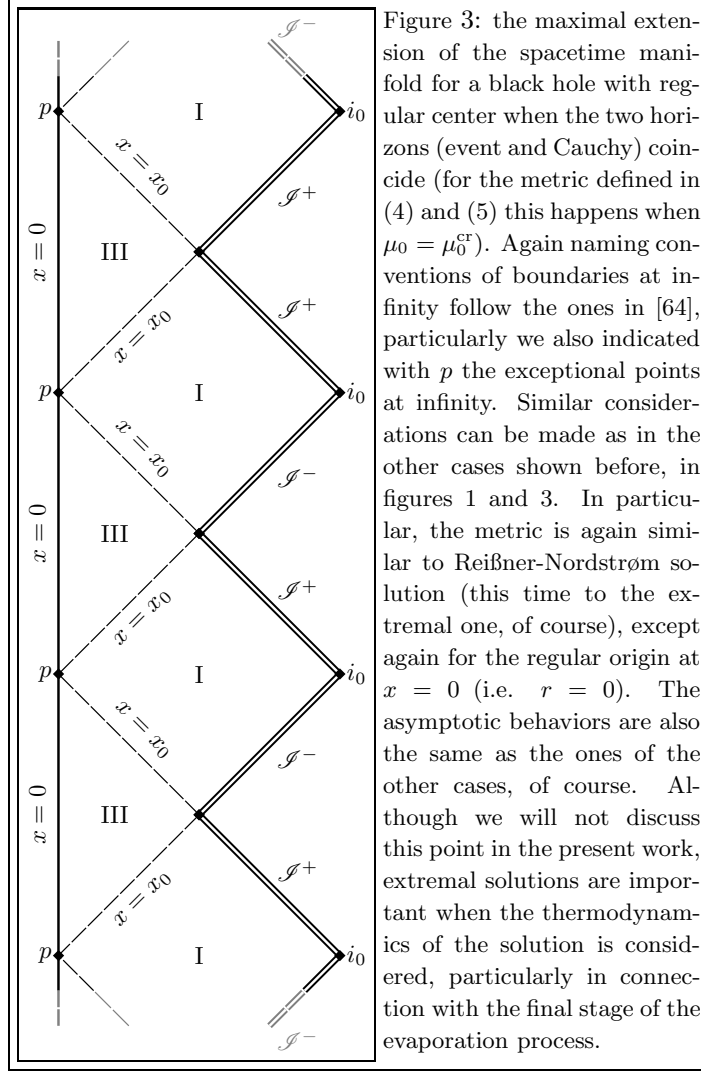
and will violate the strong energy condition<sup>15</sup> somewhere.

3. When the solution has two horizons, the larger one is an event horizon, whereas the smaller one is a Cauchy horizon, so that the solution is not globally hyperbolic (see, again, figure 1).
4. The hypothesis about asymptotic flatness can be relaxed, to allow the presence of a cosmological constant; this aspect has also been extensively discussed [40, 35, 25].

Other general results about solutions with a de Sitter core have appeared in the literature (see for instance [76], where an equation of state of the *elastic* type [77] is considered, as well as the analysis in [52]); extensions of the above results with gravity coupled to a nonlinear scalar field are also very interesting (see, e.g., [22, 50] and references therein). This general analysis is of course important and preliminary for physical application of these ideas, which also have appeared (see, for instance, [1]).

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<sup>15</sup>The strong energy condition requires  $(T_{\mu\nu} - Tg_{\mu\nu}/2)\zeta^\mu\zeta^\nu \geq 0$  for all future-directed timelike vectors  $\zeta^\alpha$  ( $T$  is the trace of  $T_{\mu\nu}$ ). This is equivalent to the requirements  $\rho + p_1 + p_2 + p_3 \geq 0 \wedge \rho + p_k \geq 0$  ( $k = 1, 2, 3$ ), where  $\rho$  is the energy density and  $p_k$  ( $k = 1, 2, 3$ ) the principal pressures; in presence of spherical symmetry this is equivalent to  $\rho + p_r + 2p_t \geq 0 \wedge \rho + p_r \geq 0 \wedge \rho + p_t \geq 0$ , where  $p_r$  is the radial pressure and  $p_t$  the tangential pressure.





## 2.6 Synopsis

We have, thus, reviewed that black hole with a regular center can be characterized, in a very precise way, in the context of general relativity; solutions which are spherically symmetric, everywhere regular, satisfy the weak energy condition and are asymptotically Schwarzschild have known global structures (as in figures 1, 2 and 3); if they have an event horizon, they must have a Cauchy horizon (and they admit the limit in which the two coincide): in these cases they are not globally hyperbolic. Non-essential details of particular solutions are specified by a unique function, the energy density, which has to be a non-increasing function tending to zero fast enough at large distance. In all generality the matter source will be an anisotropic fluid, which realizes a radial vacuum structure and violates somewhere the strong energy condition. We feel to remark the nice *travel of knowledge* that, starting from the earliest idea about the state of matter at very high densities, could find its way in the thin space left open by singularity theorems, to obtain some particular realization of singularity avoidance, first, and their general characterization, later.

## 3 Gaussian sources

Recently another neutral realization of a black hole with a regular center has been derived [93] and then generalized to the charged case [2]. With the notation that we set up above, the neutral solution can be described by the following choice of the metric function:

$$f(r) = 1 - \frac{2m(r)}{r}, \quad m(r) = \frac{2m_0}{\pi^{1/2}} \gamma\left(\frac{3}{2}, \frac{r^2}{4\Theta}\right). \quad (3)$$

We will restrict ourself to the solution interpreted as a solution of classical Einstein equations without entering into a discussion of the, also interesting, motivations for it, which will be reviewed elsewhere by one of the authors of [93]. For our subsequent analysis it is enough to remember that the metric in equation (3) is obtained assuming a Gaussian shaped, radial, mass energy density with total mass  $m_0$  and width proportional to  $\sqrt{\Theta}$ : the property  $T_t^t = T_r^r$  is also assumed, so that from the above results we know that the metric is a realization of a black hole with regular center.

To streamline the following analysis, we will preliminarily make the change of variables  $r = \sqrt{2\Theta}x$ ,  $t = y/\sqrt{2\Theta}$ : in the coordinate system  $(y, x, \vartheta, \varphi)$  the metric, then, takes the form

$$g_{\mu\nu} = \text{diag}\left(-f(x)/2\Theta, 2\Theta f(x)^{-1}, 2\Theta x^2, 2\Theta x^2 \sin^2 \vartheta\right) \quad (4)$$

where

$$f(x) = 1 - \frac{2\mu(x)}{x}, \quad \mu(x) = \frac{2\mu_0}{\pi^{1/2}} \gamma\left(\frac{3}{2}, \frac{x^2}{2}\right) = \frac{m(r(x))}{\sqrt{2\Theta}}, \quad \mu_0 = \frac{m_0}{\sqrt{2\Theta}}. \quad (5)$$

Note that, with the metric in this form,  $x$  does not represent the circumferential radius, which is instead given by  $\sqrt{2\Theta}x$ . Moreover, both  $x$  and  $\mu_0$  are

dimensionless parameters and  $\mu(x)$  is a dimensionless function. The total mass energy seen by an observer at asymptotic infinity is  $m_0 = \sqrt{2\Theta}\mu_0$ , whereas  $m(r) = \sqrt{2\Theta}\mu(x)$  is the total mass energy inside the radius  $r = \sqrt{2\Theta}x$ .

With the intent of making some acquaintance with the parameters of the model, we will reproduce here some general results for the metric in terms of the dimensionless set up described above. These properties can be analyzed directly from the study of the dimensionless metric function  $f(x) = 1 - 2\mu(x)/x$  and can be conveniently expressed in terms of the two parameters  $\mu_0$ , already defined in (5), and  $x_0$ , which is the only positive root of the equation<sup>16</sup>

$$2e^{-x^2/2} \frac{x}{\sqrt{2}} = \frac{\gamma(3/2, x^2/2)}{x^2/2}.$$

In particular,

$$f'(x) = 0 \quad \Leftrightarrow \quad x = x_0 \vee x = 0$$

and

$$f''(0) < 0, \quad f''(x_0) > 0$$

so that  $x = 0$  is a local maximum and  $x = x_0$  is a local minimum. At these points the values of the metric function are

$$f(0) = 1, \quad f(x_0) = 1 - \frac{\mu_0}{\mu_0^{\text{cr}}}, \quad \text{where} \quad \mu_0^{\text{cr}} = \frac{\pi^{1/2} e^{x_0^2/2}}{4\sqrt{2} x_0^2/2};$$

from the additional result  $\lim_{x \rightarrow +\infty} \mu(x) = 1^-$ , which follows remembering the definition of  $\mu(x)$  in (5) and the identity

$$\lim_{y \rightarrow +\infty} \frac{2}{\pi^{1/2}} \gamma\left(\frac{3}{2}, y\right) = \frac{2}{\pi^{1/2}} \Gamma\left(\frac{3}{2}\right) = 1, \quad (6)$$

it turns out that three cases can happen:

$\mu_0 > \mu_0^{\text{cr}}$  the metric function  $f(x)$  vanishes at two points  $x_1 < x_2$ ;

$\mu_0 = \mu_0^{\text{cr}}$  the metric function has a double zero, since  $f(x_0) = 0$  exactly at the minimum point  $x_0$ ;

$0 \leq \mu_0 < \mu_0^{\text{cr}}$  the metric function is always positive, so it has no zeroes.

In the case in which the metric function has two zeroes, clearly we have  $0 \leq x_1 \leq x_0$ . To locate (slightly) more precisely the position of  $x_2$ , we remember the definition of  $\mu(x)$  in (5), the property (6) and that the lower incomplete gamma function is a monotonically non-decreasing function of its second argument (since  $d_y \gamma(3/2, y) = \sqrt{y} e^{-y} \geq 0$ ). Thus  $\mu(x) \leq \mu_0$  and the difference between  $\mu(x)$  and  $\mu_0$  decreases to zero as  $x \rightarrow \infty$ . Thus

$$\lim_{\mu_0 \rightarrow +\infty} \frac{x_2}{2\mu_0} = 1$$

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<sup>16</sup>The existence of one and only one positive root comes from the general results discussed in subsection 2.5, given that the equation below is equivalent to  $f'(x) = 0$ ; anyway, this property can also be easily verified directly.

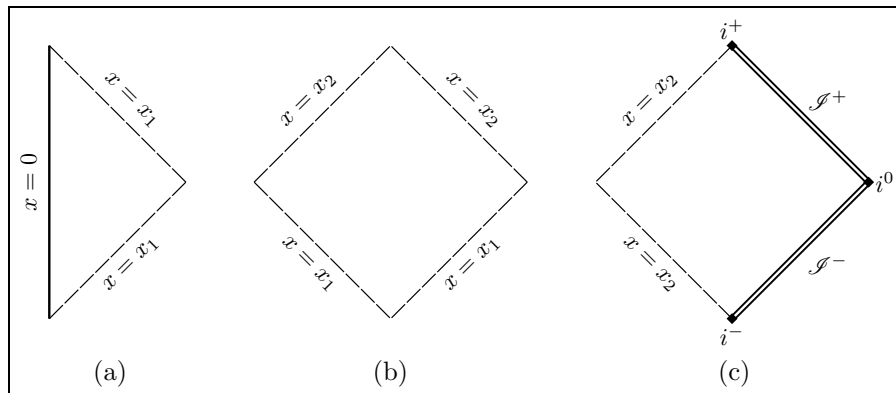


Figure 4: blocks to be joined to obtain the maximal extension of the metric in (4), (5), when we have  $\mu_0 > \mu_0^{\text{cr}}$ . Three blocks can be constructed, corresponding (a) to a timelike region between  $x = 0$  and  $x = x_1$ , (b) to a spacelike region between  $x = x_1$  and  $x = x_2$  and (c) to another timelike region between  $x = x_2$  and infinity. See also the main text.

and always  $x_2 \leq 2\mu_0$ ; in other words  $2\mu_0$  is an upper bound for the position of  $x_2$ . Then, in general

$$0 \leq x_1 \leq x_0 \leq x_2 \leq 2\mu_0.$$

A numerical evaluation gives  $x_0 \approx 2.1372$  and  $\mu_0^{\text{cr}} \approx 1.3464$ . For future reference, we also remember that the characteristic scale  $r \sim \sqrt{2\Theta}$ , corresponds to  $x \sim 1$ . The above results are a dimensionless paraphrase of the one present in [93], which will turn out to be convenient later on; they realize in the present particular case the general framework described in subsection 2.5.

### 3.1 Maximal extension

Using the properties detailed above, we will now reproduce in this particular case the general results about the maximal extension of the solution. Since the time-like two surface that we obtain ignoring the spherically symmetric component,  $\mathbb{S}^2$ , of the spacetime manifold satisfies all the requirements of the diagrammatic extension procedure described in ref. [130] we are going to follow this immediate procedure. Three cases have then to be analyzed, corresponding to the presence of two, one or no zeroes of the metric function  $f(x)$ . Conventions for the labels of boundaries, which in this case are infinities, follow ref. [64], but note that a double line represents asymptotically flat null infinity.

**Metric function with two zeroes at  $x_1$  and  $x_2$ .** In this case the block extension to obtain the full spacetime manifold goes as follows. The two zeros divide the  $[0, +\infty)$  interval of variation of the  $x$  coordinate in three regions. The block corresponding to the first of them is the one labelled (a) in figure 4 and

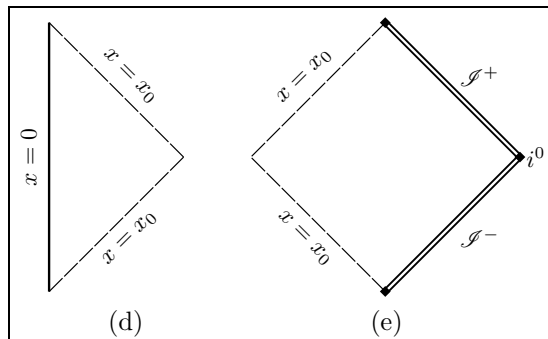


Figure 5: blocks to be joined to obtain the maximal extension of the metric (4), (5) when we have  $\mu_0 = \mu_0^{\text{cr}}$ . Two blocks can be constructed, corresponding to a timelike region between  $x = 0$  and  $x = x_1$  (d) and another timelike region between  $x = x_1$  and infinity (e). The maximally extended manifold is shown in figure 3 on page 16.

covers the region between  $x = 0$  and  $x = x_1$ ;  $x = 0$  is timelike and constant- $x$  lines join the bottom vertex with the top one. A second block (labelled (b) in figure 4) covers the region between  $x_1$  and  $x_2$ . Now constant  $x$ -curves are in the opposite direction as compared with the previous block, since the value of the metric function in this region is non-positive, which corresponds to exchanging the timelike and spacelike directions: they thus join the left and right vertices of the corresponding block. Finally the third block, labelled (c) in figure 4, covers the region between  $x_2$  and infinity, has again  $f(x) \geq 0$  and the constant- $x$  lines are directed as in the first diagram, so they join  $i^-$  and  $i^+$ . The metric is asymptotically flat and we represent this fact in the diagram with the doubled lines at null future and past infinity. The maximally extended spacetime is obtained by sewing together the blocks described above [130]. This gives the result in figure 1 on page 7. The global extension is Reissner-Nordström like, but note that  $x = 0$  is not a singularity. A point in the diagram, at position  $x$ , represents a 2-sphere  $\mathbb{S}_2$  of radius  $\sqrt{2\Theta}x$ . Dashed lines represent values of  $x$  at which  $f(x) = 0$  and they are null lines. The maximal extension has been, in fact, conformally mapped so that light rays travel along lines forming  $\pm 45$  degrees with the horizontal direction.

**Metric function with one zero at  $x_0$ .** Although the result will not be a surprise, for completeness we analyze also this case, which corresponds to  $\mu_0 = \mu_0^{\text{cr}}$ ; again we find that the maximally extended diagram is of the Reissner-Nordström type (extremal, of course) and, again, with a regular origin at  $x = 0$ . The extension procedure can be graphically performed as before. The relevant blocks are now two, shown in figure 5. The first of them (labelled (d) in figure 5) covers the patch between  $x = 0$  and  $x = x_0$ : at  $x = x_0$  the metric function and its first derivative vanish (dashed line). The second (labelled (e) in figure 5) covers the patch between  $x = x_0$  and infinity, which is asymptotically flat

(the double lines are, as before past and future null infinity). Combining this diagrams in all possible ways, so that the metric in the resulting manifold is regular across the sews, gives, as a final result, the maximal extended diagram in Fig. 3. Again, this is a well know infinite sequence of the patches described above, except for the fact that  $x = 0$  is regular (concerning the structure of the diagram and the representation of light cones, the same technical details apply, as in the previous case of Fig. 1).

**Metric function with no zeros.** We deal quickly with the last case, which corresponds to<sup>17</sup>  $0 < \mu_0 < \mu_0^{\text{cr}}$ . Now the metric never vanishes and only one block is obtained, which also represents the maximal extension of the metric in this range of variation of  $\mu_0$  (the result is as in figure 2). The origin is again regular, so that we obtain a Minkowski-like structure, except that spacetime is not empty in this case. If we consider this result in conjunction with the two previous cases, we can complete the analogy with the Reissner-Nordström solution that we repeatedly emphasized: note that also in this case we have a regular origin instead that the Reissner-Nordström timelike, naked singularity.

We have thus reproduced in this particular realization the general result discussed in subsection 2.5, i.e. the fact that the maximal extension, geometrically, “looks like” the Reissner-Nordström solution but with a regular origin. This, non minor, difference makes the spacetime manifold everywhere well behaved. After this *detour* to explicitly discuss the global geometry of the spacetime, we will turn to its energy matter content in the following section.

### 3.2 Energy conditions

Up to now we have discussed geometrically the spacetime associated with the metric (4), (5) as a representative of the more general class of black hole spacetime with a regular center, without any reference to the matter content that generates the solution. We will briefly review here the properties of the sources; since the metric (4), (5) has been derived elsewhere, we will take here the fast way, and just calculate the Einstein tensor associated to the solution. This is given by

$$G_\nu^\mu = \frac{1}{\sqrt{2\Theta}} \text{diag} \left( -\frac{2\mu'(x)}{x^2}, -\frac{2\mu'(x)}{x^2}, -\frac{\mu''(x)}{x}, -\frac{\mu''(x)}{x} \right); \quad (7)$$

so that the corresponding stress energy tensor is

$$T_\nu^\mu = \frac{1}{8\pi} G_\nu^\mu = \frac{1}{8\pi\sqrt{2\Theta}} \text{diag}(-\epsilon(x), p_x(x), p_t(x), p_t(x)), \quad (8)$$

where  $\epsilon(x)$  is the energy density,  $p_x(x)$  the radial pressure and  $p_t(x)$  the tangential pressure. Explicitly,

$$\epsilon(x) = \frac{2\mu'(x)}{x^2} = \frac{\mu_0}{\sqrt{2\Theta}(2\pi)^{3/2}} e^{-x^2/2}$$

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<sup>17</sup>We are excluding the case  $\mu_0 = 0$ , which corresponds to Minkowski spacetime.

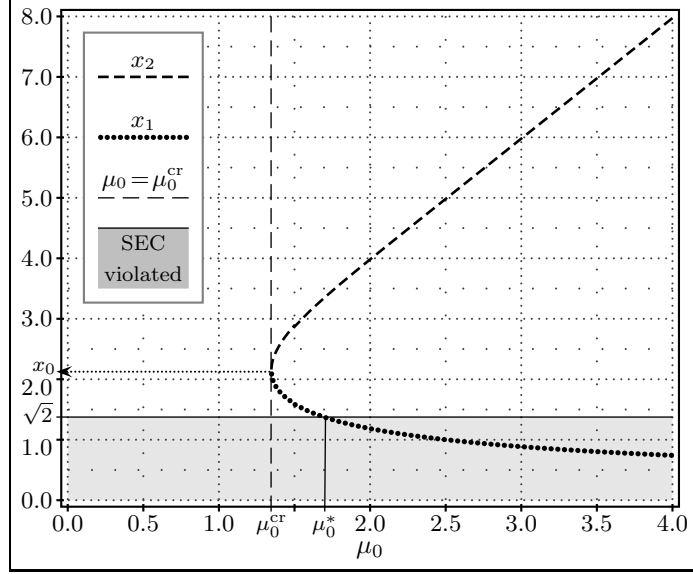


Figure 6: plot i) of the position of the two zeroes,  $x_1$  and  $x_2$ , of the dimensionless metric function and ii) of the position of the dimensionless radius below which the strong energy conditions is violated as functions of  $\mu_0$  (the second quantity is constant in the chosen units and corresponds to the horizontal line with ordinate  $\sqrt{2}$ ). The grayed area shows where the strong energy condition is violated. This happens a) for the solutions of the G-lump kind, i.e. the ones with no horizons which have  $0 < \mu_0 < \mu_0^{\text{cr}}$ , b) for the extremal solutions inside the degenerate horizon and for the two horizon solutions c1) inside the Cauchy horizon or c3) between the two horizons or c2) until the inner Cauchy horizon, depending on the value of  $\mu_0$  and if it is smaller, bigger or equal than  $\mu^*$  (the itemization of the various possibilities follows the one in the main text).

$$\begin{aligned}
 p_x(x) &= -\frac{2\mu'(x)}{x^2} = -\epsilon(x) \\
 p_t(x) &= -\frac{\mu''(x)}{x} = -\frac{\mu_0}{\sqrt{2}\Theta(2\pi)^{3/2}} e^{-x^2/2} \left(1 - \frac{x^2}{2}\right).
 \end{aligned} \tag{9}$$

From the above results it is easy to see that *the weak energy condition is always satisfied*, since

$$\epsilon(x) \geq 0, \quad \epsilon(x) + p_x(x) = 0 \geq 0, \quad \text{and} \quad \epsilon(x) + p_t(x) \sim x^2 e^{-x^2/2} \geq 0.$$

This again matches with the general properties of black holes with a regular center discussed in subsection 2.5. We are now interested to consider the strong energy condition, which we know has to be violated, to determine the qualitative properties of this violation. We would like, in particular, to identify where the

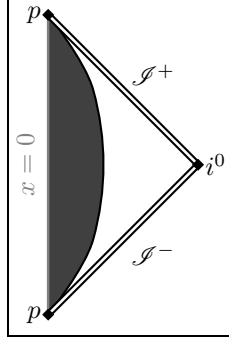


Figure 7: qualitative representation in the maximally extended diagram of a solution with no horizons (i.e with  $0 < \mu_0 < \mu_0^{\text{cr}}$ ) of the region where the strong energy condition is violated (dark gray area). In this solutions a possible naked singularity (which, for instance, can be found in the geometrically analogous Reißner-Nordström case) is replaced by a strong energy condition violating region. This is case a) in the main text.

violation can occur. With pure algebra we obtain

$$\epsilon(x) + p_x(x) + 2p_t(x) = 2p(t)(x) \sim e^{-x^2/2}(x^2 - 2)$$

so that the matter which is the source in the present solution violates the strong energy condition for  $0 \leq x < \sqrt{2}$  (the choice of scale for the various coordinates allows for a very immediate expression of this result, in which no free parameters appear). A handy way to understand where, in the full spacetime manifold, the violation occurs, is to plot, at once, the position of the horizons and of the border of the strong energy condition violating region. This is the content of figure 6. The zeroes of the metric function, corresponding to the points  $x_1$  and  $x_2$  if  $\mu_0 > \mu_0^{\text{cr}}$  (and that we called  $x_0$  in the limiting case in which they coincide, i.e. when  $\mu_0 = \mu_0^{\text{cr}}$ ) are plotted as functions of  $\mu_0$ . We recognize some of the features already discussed above, as the presence of the critical value  $\mu_0^{\text{cr}}$  below which no zeroes of the metric function appear; moreover the grayed area represents the values of the dimensionless radial coordinate  $x$  at which the strong energy condition is violated. We then see that:

- a) the strong energy condition is violated “around the regular origin” for solutions which do not posses horizons (i.e. the G-lump solutions of subsection 2.4); this can be seen from the part of the diagram on the left of the vertical dashed line, i.e. for  $0 < \mu_0 < \mu_0^{\text{cr}}$  and is represented by the dark gray area in the maximally extended diagram of figure 7.
- b) in the extremal case, the strong energy condition is violated for values of  $x$  which are completely inside the region  $x < x_0$ , i.e. inside the degenerate horizon; this is transparent from figure 6 and represented, again with a dark gray shading, in the maximally extended diagram of figure 8.

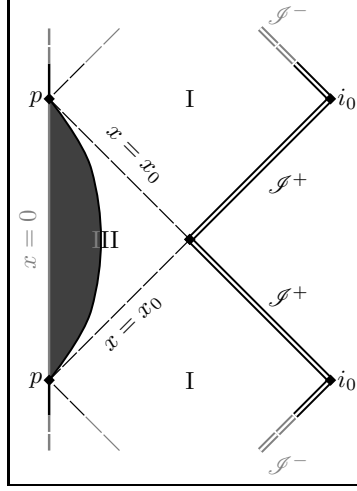


Figure 8: representation of the region in which the strong energy condition is violated in the case in which  $\mu_0 = \mu_0^{\text{cr}}$ , i.e. when we have an extremal black hole with a regular center. As seen from figure 6 the violation occurs inside the degenerate horizon  $x_0$ . The dark gray shaded region is a qualitative representation of the region of spacetime affected by this violation. This is case b) in the main text.

c) in the non-extremal case, the strong energy conditions can be violated in three qualitatively different situations, which can also be identified starting from figure 6:

- c1) in a first case, for values of  $\mu_0$  bigger than  $\mu_0^{\text{cr}}$  but smaller than  $\mu_0^*$  (we name in this way the value of  $\mu_0$  at which the inner Cauchy horizon is located at  $x = \sqrt{2}$ , i.e. at which the black dotted line crosses the constant  $\sqrt{2}$  line in figure 6) the strong energy condition violating region is completely confined inside the region  $x < x_1$ , i.e. inside the inner Cauchy horizon; this is shown in figure 9, panel c1), with a dark gray region, again;
- c2) in an intermediate case, with  $\mu_0^{\text{cr}} = \mu_0^*$ , the strong energy condition violating region is exactly the region  $x \leq x_1$ ;
- c3) in a final third case, when  $\mu_0 > \mu_0^* > \mu_0^{\text{cr}}$ , the strong energy condition violating region includes all the spacetime inside the Cauchy horizon and part of the region between the Cauchy and event horizon, up to a value of the radius that depends from the specific value of  $\mu_0$ . With the same conventions of the previous pictures this is shown in figure 9, panel c3).

Notice that the above results are independent of the scale  $\sqrt{2\Theta}$ , which, nevertheless, has to be considered when passing from the dimensionless radial coordinate



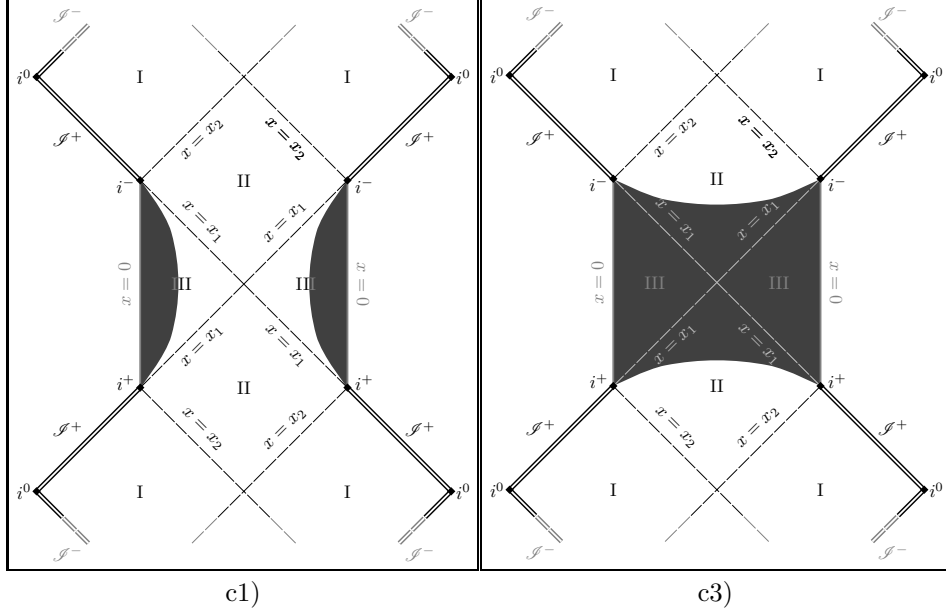


Figure 9: representation of the strong energy condition violating region in the case in which  $\mu_0 > \mu_0^{\text{cr}}$ , so that the maximally extended spacetime has the structure shown in figure 1 on page 7. As discussed in the main text, three situations can happen: only cases c1) and c3) are shown here (items have the same name as in the discussion of the main text). Case c1) corresponds to the case  $\mu_0^{\text{cr}} < \mu_0 < \mu_0^*$  and the strong energy condition is violated in a region completely contained inside the cauchy horizon. In the case c3), i.e. for  $\mu_0 > \mu_0^*$ , the strong energy condition violating region includes, instead, all the spacetime inside the Cauchy horizon and part of the region between the Cauchy horizon and the event horizon. The in between c2) case (not shown) takes place when the strong energy condition is violated exactly inside the Cauchy horizon region. As in the previous diagrams, follow the convention to show in dark gray color the part of spacetime where the strong energy condition is violated: the representation is qualitatively faithful.

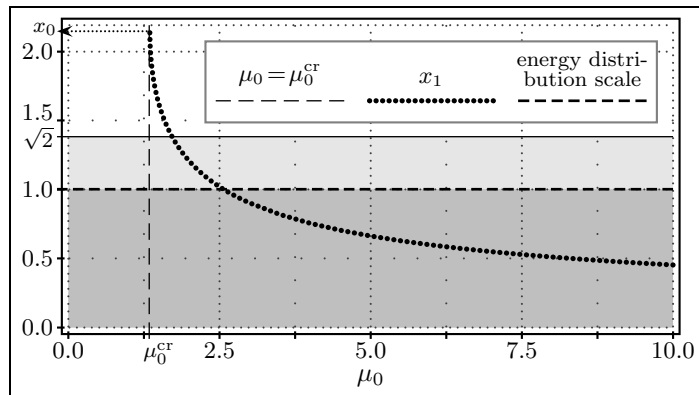


Figure 10: this diagram compares, for different values of  $\mu_0$ , the position of the Cauchy horizon, the size of the region violating the strong energy condition (*union* of light and dark gray area) and the region corresponding to the characteristic size of the Gaussian energy distribution which is the source of the solution (horizontal dashed line). We see that the size of the energy violating region and the characteristic size of the energy distribution are of comparable magnitude and that, both, extend beyond the Cauchy horizon for large enough values of  $\mu_0$ .

$x$  to physical ones, as for instance the circumferential radius  $r$ . The analysis identifies the region of spacetime where the violation of the strong energy condition occurs. It is also interesting to analyze where this violation occurs with respect to the characteristic scale of the energy density distribution; let us take the characteristic scale to be at  $r \sim \sqrt{2\Theta}$ . This corresponds to  $x \sim 1$ ; we would like to compare this value with the maximum distance at which the strong energy condition is violated, a consideration that can be easily made looking at figure 10, where, together with the inner Cauchy horizon, the strong energy condition violating region (the gray area, i.e. *union* of light and dark gray areas) and the region occupied by the characteristic width of the energy distribution (dark gray area) are emphasized. We can, then, see that the energy violating region is of the same size (as order of magnitude) of the characteristic width of the Gaussian energy distribution. For large enough values of the dimensionless mass  $\mu_0$  both regions extend beyond the inner Cauchy horizon, but none of them ever occupies space outside the event horizon.

### 3.3 Discussion

In this section we have seen that a recently obtained solution of Einstein equations [93, 2] is, in fact, a spherically symmetric black hole with regular center. Using the general results of subsection 2.5 this can be understood directly from the choice of the Gaussian energy density source and from the assumptions on the stress energy tensor: nevertheless we pedagogically derived all the stan-

dard properties of the solution. In view of applications of this model to phenomenological situations, it is important to study where the energy conditions are violated. From general arguments the weak energy condition is satisfied, but the strong one must be violated. A violation of the strong energy condition is equivalent to a violation of the attractive character of gravity, and must be confined to scales where it can be physically explained, perhaps using non classical effects.

We have seen that when horizons are present the violation always takes place inside the event horizon, but the non singular character of the metric and the fact that it is not globally hyperbolic make this result, by itself, not very significative: if an observer *could* travel (we ignore in the following qualitative description stability issues) from one asymptotic region I to another (see figure 1) crossing regions II, III and II again, in this order, then (s)he might directly experience a violation of the strong energy condition. In this respect, situations in which  $\mu_0 > \mu^*$ , shown in figure 9 panel c3) could present bigger troubles from the point of view of the phenomenological interpretation, whereas cases like the one in panel c1) in the same figure appear more sound: it thus seems that, quite generically, natural upper bounds for  $\mu_0$  can be obtained. Given that  $\mu_0 = m_0/\sqrt{2\Theta}$ , we then see that pushing  $\sqrt{2\Theta}$  to lower and lower values requires the total mass to be also smaller and smaller, *if we insist that the strong energy condition violation has to take place inside the Cauchy horizon*. At the same time, we point out that pushing too far away the scale at which the strong energy condition is violated, would provide us with a solution which could be hardly distinguishable from other ones: this suggest that for phenomenological applications it may be important to carefully discuss the various scales present in the problem keeping in mind, both, the global causal structure as well as the strong energy condition violating region. After this note of caution, we think that it is interesting to observe how the violation of the energy condition *does* take place on scales comparable to the scales which characterize the matter-energy distribution; this fact is then suggestive, since, if non classical effects are advocated as motivations for the matter-energy source, then we see that they actually modify the spacetime structure mostly at the scale at which they become relevant: this is, in our opinion, an important consistency check *when dealing with non linear situations*.

Although the configurations *with horizon* are the most studied and emphasized ones, we think that the G-lump solutions are also interesting and, perhaps, closer in spirit to what we would intuitively think as a regular spacetime, since they do not have horizons or other structural/apparent pathologies. Notice that G-lump solutions are realized for  $\mu_0 < \mu_0^{\text{cr}}$ , which, after fixing the scale  $\sqrt{2\Theta}$ , can be traduced into an upper bound for their total mass-energy  $m_0$ , consistently with their localized structure (see also the discussion in [35]).

It is very likely that similar considerations will hold also for other spherically symmetric black holes with a regular center: it then becomes interesting to discuss if/how different realizations could be practically distinguished, but we will not discuss this point here.

## 4 Conclusions

Summarizing, we have reviewed various spherically symmetric black hole solutions with a regular center: in the first part of this contribution we kept an historical perspective, and we tried to emphasize alternative approaches and ideas, showing at the end that they can be all understood inside the same general framework. Then, in the second part, we specialized to a very specific solution and we showed how it also fits the above mentioned general framework. We analyzed in detail the violation of the strong energy condition: results similar to the one that we have shown here are likely to hold for different specific realizations. If we, thus, interpret the non standard behavior of these solutions at small scales invoking some sort of non-classical effects, it is remarkable that, self consistently, these proposals violate known properties of matter and energy only at scales at which these effects are supposed to be dominant. This might be *not granted* in a *non linear* theory and, whether we consider the G-lump or the black hole structures, it would be interesting to obtain these solutions in a consistent, fully non-classical treatment of spacetime and its matter fields content, or, preliminarily, at least incorporate, effectively, this effects also in the left hand side of Einstein equations.

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